41000 Graph Theory, Ph.D.

# Student Talk: A view on Percolation and Spin Systems

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## Preamble

> A mathematician is a device for turning coffee into theorems.
>
> Alfréd Rényi, often ascribed to Paul Erdős

This document is a short 1 hour lecture on percolation. Due to its variety, complications and potential, it would be an understatement to say it is a beautiful subject.

As a physical phenomenon, percolation describes a liquid passing through a filter. Perhaps due to its implementation in coffee percolators, it has been of great interest for mathematicians, as the famous quote above funnily points out.

In a more scientific sense, it turns out to be a model that has wide applications in network theory, with the purpose of assessing reliability of graphical structures, as well as their arrangement and aspect. Simultaneously, it spurred innovation on the theoretical side, which is the main interest of this production.

**Aims** The purpose of this lecture is to draw connections between a probabilistic-graphical and a Statistical Mechanics model. The former is very similar to the ones found in Graph Theory, and the latter to Statistical Physics models. Both were subjects explored in Ph.D. courses held at Bocconi University[1]. As we will see, in a wider sense, the two topics have strong similarities. Speaking about them separately is a mistaken excuse to motivate the choice and place this lecture in between.

**Time constraints and materials** The difficulty of the subject forces us to just have an overview. The topic is wide and very technical. To have a beginners' understanding, I needed to spend some time reading and doing the proofs, something that cannot be done in a single lecture. Thus, I will restrict the content to intuitions, hopefully some proofs and partial results in the span of around 10 pages[2]. To accompany the speech with delightful

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[1] link to GT syllabus and link to SP syllabus

[2] I will skip some parts which were already discussed. A 1h30′ lecture in the two courses was of this length.

images, a presentation with simulations impossible to draw on a blackboard will be used. All of the images in this document will be replicated there.

**Sources** The flavour of exposition is based on a partial extraction of the framework of the recent 2022 Fields medalist Hugo Duminil-Copin, who strongly contributed to the recent advancements in the field. More precisely, as a novice I read (more or less in order) the following publications:

- *"Sixty years of Percolation"* [Duminil-Copin, 2017c]
- *"Introduction to Bernoulli Percolation"* [Duminil-Copin, 2017a]
- *"Lectures on the Ising and Potts models on the hypercubic lattice"* [Duminil-Copin, 2017b]
- *"A New Proof of the Sharpness of the Phase Transition for Bernoulli Percolation and the Ising Model"* [Duminil-Copin and Tassion, 2016]

I would like to stress that no contributions are added on my side. Additionally, I made use of Grimmet's book [Grimmett, 1999], two Ph.D. thesis [Li, 2017; Duminil-Copin, 2011] and a description in the simplest settings of the results of the last item in the list [Duminil-Copin and Tassion, 2017].
Whenever possible, I will try to redirect the reader to the right citation, but please note that being a lecture this work will be most of the times avoided. For an accessible review, the first reading [Duminil-Copin, 2017c] is exhaustive.

# 1 Intro to Percolation and Spin Systems

## 1.1 Notation

Throughout the document, we use the following basic objects:

- $\|\cdot\|$ is the Euclidean norm
- arbitrary graph: $G = (V, E)$, where $E \subset V \times V$ is an unordered collection of edges denoted as $e = xy \in E$
- $d$-dimensional lattice $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ where $\mathbb{V} = \mathbb{Z}^d$ and $\forall e = xy$ it holds that $\|x - y\| = 1$. Namely, an infinite grid with squares of size 1. Often only $\mathbb{Z}^d$ is used to refer to it.
- the $n$-sized square in $d$-dimensions is denoted as $\Lambda_n = [-n, n]^d \quad \forall n \geq 1$
- vertex boundary of a subgrid $G \subset \mathbb{G}$ is $\partial G = \left\{x \in V : \exists y \in \mathbb{Z}^d : xy \in \mathbb{E} \setminus E\right\}$

- connection symbol $\longleftrightarrow$, which indicates that there exists a path of edges which connects the two endpoints. For vertices we write $x \longleftrightarrow y$. Similarly when the path is restricted to being made of edges in a specific set $S$ we write $x \overset{S}{\longleftrightarrow} y$. The notion naturally extends to sets at the endpoints.
- connection to infinity is expressed as $x \longleftrightarrow \infty$ and means that the vertex $x$ belongs to an infinite cluster identified by the infinite connection. Here by cluster we mean a maximal connected component.

## 1.2 Percolation

A Graph can be seen as a set of sites and connections represented by edges. In this perspective, passage of a fluid is possible in presence of a connection between vertices through a path. We formalize this notion below and present the first model.

**Definition 1.1** (Percolation Configuration)**.** Given a graph $G$, a function

$$\omega : E \to \{0,1\} \quad s.t. \quad e \to \omega_e = \begin{cases} 1 & open \\ 0 & closed \end{cases},$$

is said to be a percolation configuration.

**Example 1.2** (Deterministic Percolation Configuration)**.** *The following is an anti-example with the purpose of having an intuition on potential applications. We have a clique of* 5 *vertices. A percolation configuration could be:*

$$\omega_e = \begin{cases} 1 & \text{odd-odd or even-even index} \\ 0 & \text{else} \end{cases}.$$

*Figure 1 is a depiction. Suppose that the clique is a fundamental network that must preserve*

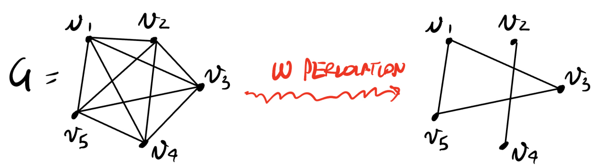


Figure 1: Anti-percolation

*some notion of reliability, and that the percolation configuration is the remaining connections after some damage (i.e. a portion of the edges have been closed by an adversary). The main question is understanding under which conditions reliability is granted.*

As a matter of fact, working with deterministic tweakings of an original graph is not satisfactory. In order to obtain bounds and meaningful results, we will make the construction **probabilistic**.

**Definition 1.3** (Percolation Model, and one specific instance)**.** A distribution over the possible $\omega$ of a given $G$. In particular, we will focus on **Bernoulli Percolation**, for which configurations are of the form:

$$\omega_e \overset{iid}{\sim} \mathcal{B}ern(p) \quad \omega_e = \begin{cases} 1 & wp \; p \\ 0 & wp \; 1-p \end{cases}.$$

In particular, the random model will act on the $\mathbb{Z}^d$ lattice or its subsets. The former has a probability space:

$$(\Omega, \mathcal{F}, \mathbb{P}) = \left(\{0,1\}^{\mathbb{E}}, \mathcal{F}, \mathbb{P}_p\right) \quad \mathcal{F} = \sigma\left(\{A \in \mathcal{F} : |A| < \infty\}\right) \quad \mathbb{P}_p = \frac{1}{Z_{\mathbb{G},p}} \prod_e \mu_e^{\mathcal{B}ern}.$$

**Notation:** for its expectation, we write $\mathbb{E}_p$. Notice that the only parameter is $p$, and that it is local.

The natural question that comes to mind is rather **global** and refers to the connectivity properties of the configurations in the probability space.

Main Question

**(MQ)** Is the model such that there is an infinite cluster?

Being an event, the first sanity check would be ensuring that such event is measurable.

**Lemma 1.4.** $\{\exists \infty \; cluster\} \in \mathcal{F}$.

*Proof.* By direct computation:

$$\{\exists \infty \; cluster\} = \bigcup_{x \in \mathbb{Z}^d} \{x \longleftrightarrow \infty\} = \bigcup_{x \in \mathbb{Z}^d} \bigcap_{n \in \mathbb{N}} \underbrace{\{\exists v : \|v - x\| = n \wedge v \longleftrightarrow x\}}_{\in \mathcal{F}} \in \mathcal{F}.$$

By countable unions and intersections. □

With this fashion, and the sole dependence on the local connection parameter $p$, we are invited to define a quantity to eventually study.

**Definition 1.5** (Origin Parameter for infinite Cluster)**.**

$$\theta(p) := \mathbb{P}_p \left[0 \longleftrightarrow \infty\right].$$

*Remark* 1.6. Notice that we are not restricting ourselves to the uniqueness of an infinite cluster, as there may be many of them. Additionally, the choice of the origin is for reference, and the infinite cluster definition would work for any starting point $x$. We are indeed exploiting invariance of $\mathbb{Z}^d$ under translations.

Numerical simulations on the lattice suggest that there exists a critical probability such that:

- below such threshold there are no infinite clusters
- above there is an infinite cluster

For a visualization, see Figure 2.

Figure 2: Percolation simulation at $p < \frac{1}{2}, p = \frac{1}{2}, p > \frac{1}{2}$ in the square lattice. Source [Li, 2017]

It is worth noticing that in principle we are only able to assess the loose verification of such property since simulations are at finite lattices (for obvious reasons). Nevertheless, we are motivated to define a critical probability for the infinite size lattice and attempt to derive formal results about its behavior.

**Definition 1.7** (Critical Probability)**.**

$$p_c := \inf \{p \in [0,1] : \theta(p) > 0\} .$$

Motivated by this construction we could loosely define three other important questions for lattices:

Sub questions

**Q1** is $\theta_p$ continuous?

**Q2** is it monotonic?

**Q3** is it just positive or does it become 1 above criticality (i.e. almost surely an infinite cluster exists)

Two of the most important objects for such objectives are reported below.

**Definition 1.8** (Partial order on configurations)**.** The binary assignment for each edge can be used to formalize a notion of magnitude in the following sense.

$$\omega \le \omega' \iff \omega_e \le \omega_{e'} \quad \forall e \in \mathbb{Z}^d.$$

This is very useful to have a concept of increasing in the $\Omega$ space.

**Definition 1.9** (Increasing events)**.** An event $\mathcal{A}$ is increasing when it includes all the greater configurations of its elements according to the partial order above. Mathematically:

$$\mathcal{A} \in \mathcal{F} : \quad \omega \in \mathcal{A}, \omega \le \omega' \implies \omega' \in \mathcal{A}.$$

Equivalently $\mathbb{1}_A(\omega)$ is non decreasing in $\Omega$.

**Definition 1.10** (Increasing functions)**.** we generalize the notion above to functions of the form $f : \Omega \to \mathbb{R}$ which are increasing in the usual sense. Namely:

$$f \quad s.t. \quad \omega \le \omega' \implies f(\omega) \le f(\omega').$$

**Definition 1.11** (Dual of planar graph)**.** For $G = (V, E)$ planar, we define its dual $G^* = (V^*, E^*)$ with $V^*$ being the faces of $G$ and $E^*$ a collection of face-face connections whenever such spaces are neighbors on one edge in $G$.

**Example 1.12** (Dual of arbitrary graph)**.** *Notice that there might be a double edge connection between two faces if two edges are neighboring. See Figure 3.*

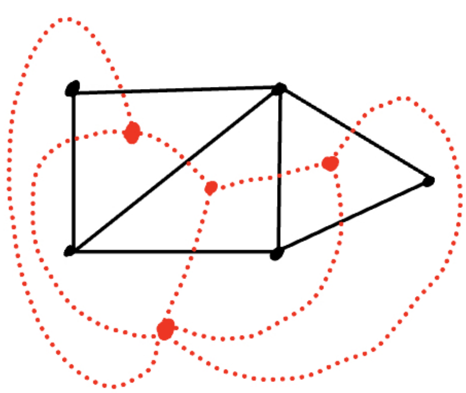

Figure 3: A dual graph

**Example 1.13** (Dual of $\mathbb{Z}^2$)**.** *The dual of the simplest lattice structure at $d = 2$ can be seen as a grid translated by a $\left(\frac{1}{2}, \frac{1}{2}\right)$ vector, with one edge in the dual for each edge in $G$.*

*When all edges are present, a pictorial representation of a subset of the grid is like that of Figure 4. This bijection also shows that a percolation configuration $\omega$ corresponds uniquely*

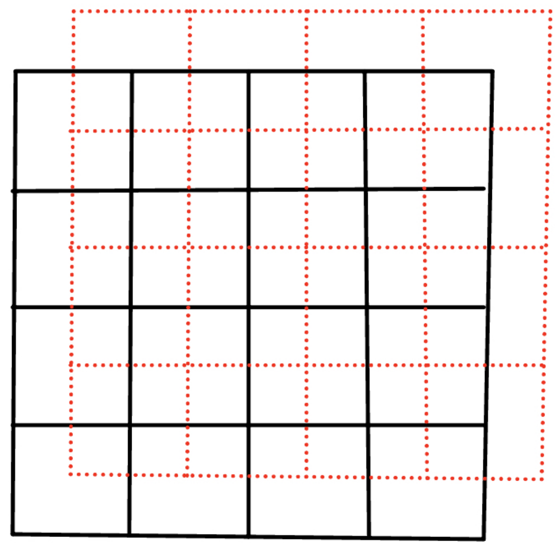

Figure 4: Dual subgrid of $\mathbb{Z}^2$

*to a percolation configuration on the dual with form:*

$$\omega^*_{e^*} = 1 - \omega_e.$$

*Therefore, open edges are closed in the dual and viceversa. Additionally, we easily conclude that if* $\omega \sim \mathbb{P}_p$ *then* $\omega^* \sim \mathbb{P}_{1-p}$ *with a space translation. An example of primal and dual percolation configurations is Figure 5.*

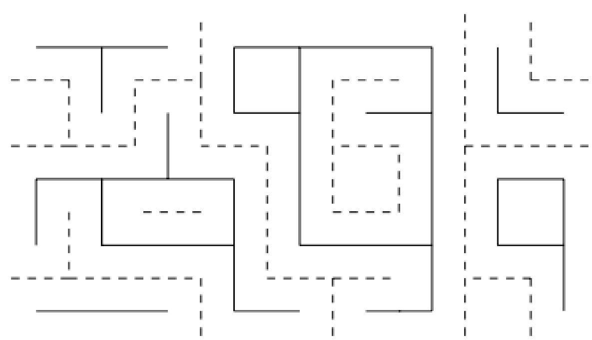

Figure 5: Primal and dual percolation. Source [Duminil-Copin, 2017a]

## 1.3 Spin Systems

In the realm of lattice models, we focus on spin systems, which are a random assignment of spins to vertices.

**Definition 1.14** (Spin variable and Spin Configuration)**.** A spin variable is an assignment to a vertex $x \in V$ of a graph $G = (V, E)$. In principle, it belongs to a set:

$$\sigma_x \in \Sigma \subset \mathbb{R}^r.$$

An assignment of spin variables for each vertex is a spin configuration:

$$\sigma = (\sigma_x : x \in V) \in \Sigma^V.$$

Again, we focus on local interactions, which in the jargon of Statistical Physics are often called **Ferromagnetic Nearest Neighbor Interactions**. An energy (Hamiltonian) notion is used to describe their macroscopic (global) behavior.

**Definition 1.15** (Free Boundary Conditions Hamiltonian)**.** For a graph $G$ with **free boundary conditions** we assign:

$$H_G^{\mathrm{f}}(\sigma) := -\sum_{xy \in E} \sigma_x \cdot \sigma_y.$$

Notice that inside the sum we have a dot product since $\sigma_x \in \Sigma \subset \mathbb{R}^r$, and that with this construction we are associating lower energy to aligned configurations. Following the maximum entropy principle, we may also assign (with some omitted adjustments in the infinite size case) a measure on the configurations.

**Definition 1.16** (Gibbs Measure)**.** Consider a graph $G$. Let $\beta \geq 0$ be an inverse temperature parameter, and $f : \Sigma^V \to R$. Assign $d\sigma = \bigotimes_{x \in V} d\sigma_x$ where $\forall x$ we have a copy of a reference measure $d\sigma_x = d\sigma_0$. The Gibbs measure of $f$ with free boundary conditions is:

$$\mu_{G,\beta}^{\mathrm{f}}[f] := \frac{\int_{\Sigma^V} f(\sigma) e^{-\beta H_G^{\mathrm{f}}(\sigma)} d\sigma}{\underbrace{\int_{\Sigma^V} e^{-\beta H_G^{\mathrm{f}}(\sigma)} d\sigma}_{=Z_{G,\beta}}}.$$

If we wish to add boundary conditions $b \in \Sigma \subset \mathbb{R}^r$ we instead force the boundary configurations to be the specified value and obtain:

$$\mu_{G,\beta}^{\mathrm{b}}[f] := \mu_{G,\beta}^{\mathrm{f}}[f \mid \sigma_x = b \; \forall x \in \partial G].$$

*Remark* 1.17. The generality of this construction allows us to identify different models depending on the choice of $d\sigma_0$ and $\Sigma$, the reference measure and the space of spin configurations.

We recover below two famous models in Statistical Physics.

**Example 1.18** (Ising Model)**.** *$d\sigma_0$ is the counting measure, $\Sigma = \{-1, 1\}$. We denote it as $\mu^{\#}_{G,\beta}$.*

**Example 1.19** ($q$ Potts Model)**.** *$d\sigma_0$ is the counting measure and $\Sigma = \mathbb{T}_q$ with $q \geq 2$, the simplex in $q$ dimensions oriented such that $e_1 = (1, 0, \dots, 0) \in \mathbb{T}_q$. In its general formalism that allows for generalization we use the simplex notions (i.e. points equidistant on a circle) and specify that its inner products are of the form:*

$$a \cdot b = \begin{cases} 1 & a = b \\ -\dfrac{1}{q-1} & a \neq b, \end{cases}$$

*which amounts to an often convenient rewriting of the interaction components $J_{xy}\delta_{\sigma_x,\sigma_y}$. We denote it as $\mu^{\#}_{G,\beta,q}$.*

Recalling that we temporarily assumed that the thermodynamic limit $G \to \mathbb{Z}^d$ made sense for the measure, we can define critical parameters that we wish to inspect for the general class of models. In particular, $\beta$ will determine the appeareance (or not) of certain behaviors in the measure. Below we report the three most important ones.

**Definition 1.20** (Critical Parameters)**.** Spontaneous magnetization:

$$\mu^{\mathrm{b}}_{\beta}[\sigma_0 \cdot b] > 0 \quad \forall b \in \Sigma, \tag{MAG$_\beta$}$$

with associated critical parameter $\beta_c \coloneqq \beta_c^{mag} = \inf\{\beta > 0 : (\mathrm{MAG}_\beta)\ \texttt{true}\}$.
Long range ordering:

$$\lim_{\|x\| \to \infty} \mu^{\mathrm{f}}_{\beta}[\sigma_0 \cdot \sigma_x] > 0, \tag{LRO$_\beta$}$$

with associated critical parameter $\beta_c^{lro} \coloneqq \inf\{\beta > 0 : (\mathrm{LRO}_\beta)\ \texttt{true}\}$.
Exponential decay of correlations:

$$\exists c_\beta > 0 \quad : \quad \mu^{\mathrm{f}}_{\beta}[\sigma_0 \cdot \sigma_x] \leq e^{-c_\beta \|x\|} \quad \forall x \in \mathbb{Z}^d, \tag{EXP$_\beta$}$$

with associated critical parameter $\beta_c^{exp} \coloneqq \sup\{\beta > 0 : (\mathrm{EXP}_\beta)\ \texttt{true}\}$.

*Remark* 1.21. We briefly interpret them as:

($\mathrm{MAG}_\beta$) boundary conditions induce alignment of the origin

($\mathrm{LRO}_\beta$) far vertices correlate with the spin at the origin

($\mathrm{EXP}_\beta$) correlation of origin and spins is exponentially small in distance

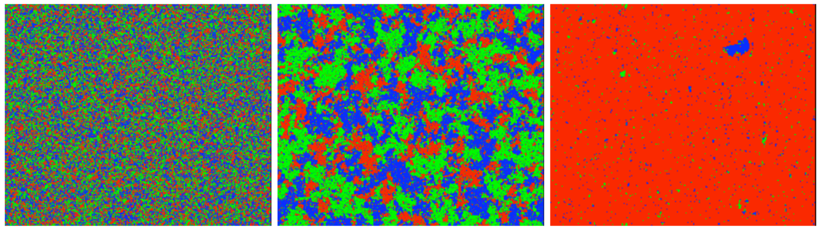

Figure 6: Simulations for $q = 3, d = 2$ at subcritical, critical, supercritical temperature. Source [Duminil-Copin, 2017b]

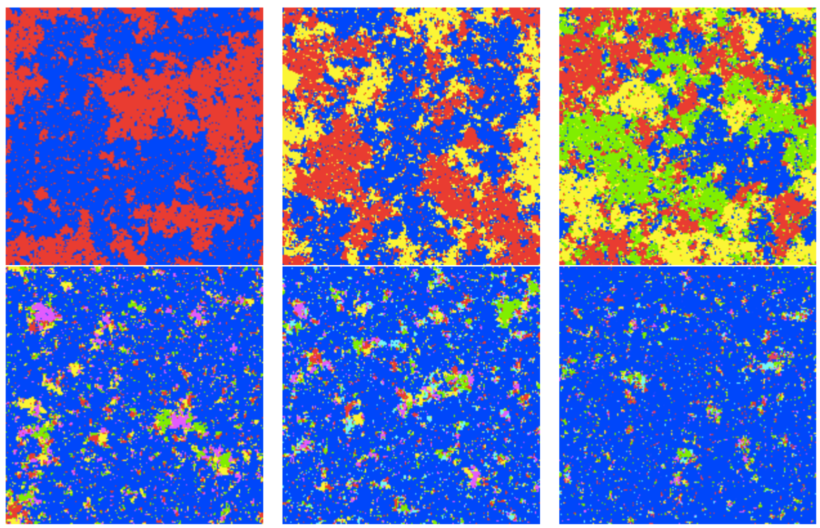

Figure 7: Simulations for $q \in \{2, 3, 4, 5, 6, 9\}$, $d = 2$ at criticality. Source [Duminil-Copin, 2017b]

Empirical simulations of the Potts model suggest that for $d \geq 2$ and $\Sigma$ discrete or $d \geq 3$ and $\Sigma$ arbitrary there exists a finite inverse temperature such that alignment $\text{MAG}_\beta$ is verified above it. This allows us to hypothesize that $\beta_c < \infty$ and motivates us to show it. In general, there are different classes of Phase Transitions. In Figures 6 and 7 we report some simulations.

**Questions for Spin systems**

**Q4** how do the three critical inverse temperature relate with each other?

**Q5** is $(\text{MAG}_{\beta_c})$ true?

## 2 The Random Cluster Model

The purpose of this section is drawing connections between percolation and spin systems. We restrict ourselves to $G \subset \mathbb{Z}^d$, with $|V| < \infty$ and edges defined by a configuration $\omega$. The number of its open edges will be $o(\omega)$, while the number of closed edges is $c(\omega)$.

**Definition 2.1** (Boundary Conditions (BC))**.** Given a graph, consider a partition of its boundary $\partial G$:

$$\xi = P_1 \cup \cdots \cup P_k : \quad \{P_i\}_{i=1}^k \textit{ partition } \partial G.$$

In practice: for vertices belonging to the same partition, merge them into a single one, keeping connections to the inner part of the graph (namely, each vertex in the partition inherits all connections of its friends). Call the induced graph $\omega^\xi$ and the number of clusters $k(\omega^\xi)$.

**Example 2.2** (Reference boundary conditions)**.** *When speaking about* ***free*** *boundary conditions, we mean performing no modifications on the percolation configuration. The symbol is* $0$*,* $\xi$ *is the set of singletons of* $\partial G$ *and* $k(\omega^0) = k(\omega)$*.*
*On the opposite*[3] *side,* ***wired*** *boundary conditions refer to using a single set for* $\xi = \partial G$*, thus forcing a connection in all vertices at the boundary. The symbol is* $1$*. In this case,* $k(\omega^1)$ *is the number of clusters if all clusters at the boundary are counted as a single one.*
*In the general case* $\xi \subset \mathbb{Z}^d$*, and two boundary vertices are in the same set* $P_i$ *if they are in the same cluster of the graph* $\xi$*.*

**Definition 2.3** (Random Cluster Model (RCM) Measure)**.** Given an edge weight $p \in [0, 1]$ a cluster weight $q > 0$ and $\xi$ boundary conditions define a measure over possible configurations of a graph $G$ as:

$$\phi^\xi_{G,p,q}[\omega] := \frac{p^{o(\omega)}(1-p)^{c(\omega)} q^{k(\omega^\xi)}}{Z^\xi_{G,p,q}}.$$

*Remark* 2.4. For $q = 1$ cluster weights have no influence on the measure and we recover the percolation measure.

The foundational property of RCMs is proved below.

**Theorem 2.5** (Domain Markov Property (DMP))**.** *More details in ([Duminil-Copin, 2011] Thm. 3.2), ([Duminil-Copin, 2017b], Sec. 1.2.1), ([Li, 2017] Prop. 1.2).*
*Consider a graph* $G$ *and a subgraph* $G'$*. For any boundary conditions* $\xi$ *and any subconfigurations* $\psi \in \{0,1\}^{E \setminus F}$ *and* $\psi' \in \{0,1\}^{E'}$ *we have that:*

$$\phi^\xi_{G,p,q}[\omega_{|E'} = \psi' \mid \omega_{|E \setminus E'} = \psi] = \phi^{\psi^\xi}_{G',p,q}[\psi'],$$

*where* $\psi^\xi$ *is the boundary condition induced on* $G'$ *by* $(\xi, \psi)$*. It is constructed by pairing vertices of* $\partial G'$ *if they are connected in* $E \setminus E'$ *by the boundary conditions* $\xi$ *on* $G$*. In loose*

[3] If we had more time, we could have proved that these two scenarios are indeed extremal on the two sides

*terms, this means that the conditional probability reduces to being equal to the measure on the subgraph with updated boundary conditions equivalent to the boundary conditions and the fixed part of the graph* $\psi$.

*Proof.* The claim holds by recursing the reasoning we will perform for $G' : E' = E \setminus \{e\}$. Let $\omega' \in \{0,1\}^{E'}$ and $\omega^e$ be the configuration such that it is equal to $\omega'$ in the subgraph and open on the missing edge. Additionally, $\xi^e$ denotes boundary conditions on $G'$ with $\xi$ and the edge $e$ open. Then for any $\omega$ we have that:

$$\begin{aligned}
\phi^{\xi}_{G,p,q}[\omega \mid \omega(e) = 1] &= \frac{\phi^{\xi}_{G,p,q}[\omega, \omega(e) = 1]}{\phi^{\xi}_{G,p,q}[\omega(e) = 1]} && \text{Def. of conditional} \\
&= \frac{\phi^{\xi}_{G,p,q}[\omega^e]}{\phi^{\xi}_{G,p,q}[\omega(e) = 1]} && \text{Notation} \\
&= \frac{Z^{\xi}_{G,p,q}}{Z^{\xi}_{G,p,q}} \frac{\prod_{f\in E} p^{\omega^e(f)}(1-p)^{1-\omega^e(f)} q^{k^{\xi}(\omega^e)}}{\sum_{\chi\in\{0,1\}^E:\chi(e)=1} \prod_{f\in E} p^{\chi(f)}(1-p)^{1-\chi(f)} q^{k^{\xi^e}(\chi)}} \\
&\text{Num: RCM definition, Den: marginalize} \\
&= \frac{p^{o(\omega)+1}(1-p)^{c(\omega)} q^{k^{\xi}(\omega^e)}}{\sum_{\chi\in\{0,1\}^E:\chi(e)=1} \prod_{f\in E'} p^{\chi(f)}(1-p)^{1-\chi(f)} p\, q^{k^{\xi^e}(\chi)}} \\
&\text{cancel normalization} \\
&= \frac{p^{o(\omega)}(1-p)^{c(\omega)} q^{k^{\xi^e}(\omega)}}{\sum_{\chi\in\{0,1\}^E:\chi(e)=1} \prod_{f\in E'} p^{\chi(f)}(1-p)^{1-\chi(f)} q^{k^{\xi^e}(\chi)}} \\
&\text{notice } k^{\xi^e(\omega)} = k^{\xi}(\omega^e) \\
&= \phi^{\xi^e}_{G',p,q}[\omega \mid_{E'}] \\
&= \phi^{\xi^e}_{G',p,q}[\omega'].
\end{aligned}$$

As mentioned above, iterating this procedure gives the claim. □

The simplest case of this property gives us for $e \in E$ and $\psi \in \{0,1\}^{E\setminus\{e\}}$ that:

$$\phi^{\xi}_{G,p,q}[\omega_e = 1 \mid \omega_{|E\setminus\{e\}} = \psi] = \phi^{\psi^{\xi}}_{\{e\},p,q}[\omega_e = 1] = \begin{cases} p & \text{if } x \overset{\psi^{\xi}}{\longleftrightarrow} y \\ \dfrac{p}{p+q(1-p)} & \text{else} \end{cases},$$

from which we also derive a nice property reported below.

**Corollary 2.6** (Finite Energy Property (FE))**.** *More details in ([Duminil-Copin, 2011] Prop. 3.3), ([Duminil-Copin, 2017b] Sec. 1.2.1), ([Li, 2017] Sec. 1.1.9).*
*Consider a graph $G$ with $p \in (0,1)$. For any boundary conditions $\xi$, configuration $\omega$ and any edge $e = xy \in E$ it holds:*

$$\exists c_{FE} > 0 \quad s.t. \quad \phi^{\xi}_{G,p,q}[\omega_e = 1 \mid \omega_{|E\setminus\{e\}} = \psi] \in [c_{FE}, 1 - c_{FE}].$$

## 2.1 Coupling

More details about this concept are found in ([Grimmett, 1999], Sec. 1.4), ([Duminil-Copin, 2011] Part II, Sec. 3), [Duminil-Copin, 2017b] Sec. 1.2.2.), ([Li, 2017] Sec. 1.1.2).
Consider $q \geq 2, q \in \mathbb{N}$, and $G$ finite. We first inspect free boundary conditions.

**Percolation to Spins:** Given $\omega \in \{0,1\}^E$ build a spin configuration $\sigma \in \mathbb{T}_q^V$ by assigning for each cluster independently a spin. For the clusters $\mathcal{C}$ in $\omega$ we are formally embedding uniformly $\sigma_{\mathcal{C}} \in \mathbb{T}_q$ with $\sigma_x = \sigma_{\mathcal{C}}\ \forall x \in \mathcal{C}$.

**Spins to Percolation:** Given $\sigma \in \mathbb{T}_q^V$ build a percolation configuration as follows:

$$\forall e = xy \in E \quad \omega_e = \begin{cases} 0 & if\ \sigma_x \neq \sigma_y \\ 1 & wp\ p\ \&\ \sigma_x = \sigma_y \\ 0 & wp\ 1-p\ \&\ \sigma_x = \sigma_y \end{cases}.$$

For an example, see Figure 8.

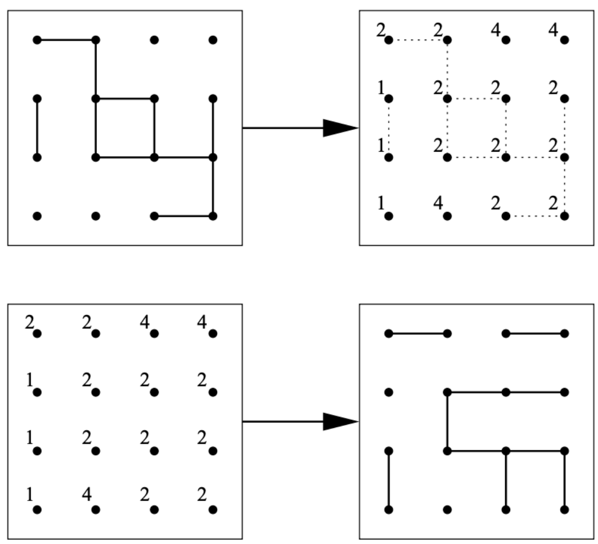


Figure 8: The methods for $q = 4$. Source [Grimmett, 1999]

**Proposition 2.7** (Coupling for Free BC)**.** *Formulated as in [Duminil-Copin, 2017b]. Assume the procedures for generating are as above. Then:*

$$\begin{aligned} &\texttt{percolation to spins} \quad \omega \sim \phi^0_{G,p,q} \implies \sigma \sim \mu^{\mathrm{f}}_{G,\beta,q}, \quad \beta := -\frac{q-1}{q}\ln(1-p), \\ &\texttt{spins to percolation} \quad \widetilde{\sigma} \sim \mu^{\mathrm{f}}_{G,\beta,q} \implies \widetilde{\omega} \sim \phi^0_{G,p,q}. \end{aligned}$$

*Proof.* We start with the first claim. Consider the joint $\boldsymbol{P}$ on $\Omega \times V$. By construction, the first marginal is $\phi^0_{G,p,q}$. The second marginal is instead what we wish to find. For this reason, introduce a compatibility notion for a pair:

$$\omega \in \{0,1\}^E, \sigma \in \mathbb{T}_q^V \quad s.t. \quad \forall xy \in E : \omega_{xy} = 1 \implies \sigma_x = \sigma_y,$$

which makes sense since if the edge is open then it must be the case that the color assigned is the same. Clearly, if $(\omega, \sigma)$ are not compatible the joint is null, i.e. $\boldsymbol{P}[(\omega, \sigma)] = 0$. If instead the pair is compatible

$$\boldsymbol{P}[(\omega, \sigma)] = \frac{1}{Z^0_{G,p,q}} p^{o(\omega)}(1-p)^{c(\omega)} q^{k(\omega)} \times \frac{1}{q^{k(\omega)}} = \frac{1}{Z^0_{G,p,q}} p^{o(\omega)}(1-p)^{c(\omega)}. \tag{1}$$

Before summing over $\omega$, we need to find a clever expression for the $\omega$ that are compatible with a given $\sigma$. Thus, for $\sigma \in \mathbb{T}_q^V$ we roll out a compatibility notion:

$$E_\sigma = \{xy \in E : \sigma_x \neq \sigma_y\} \quad s.t. \quad \omega_{xy} = 0 \; \forall xy \in E_\sigma \wedge \omega_{xy} \overset{iid}{\sim} \mathcal{B}ern(p) \; \forall xy \notin E_\sigma \quad \texttt{compatibility},$$

from which we get:

$$\begin{aligned}
\boldsymbol{P}[\sigma] &= \sum_{\omega \in \{0,1\}^E} \boldsymbol{P}[(\omega, \sigma)] \\
&= \sum_{\omega \textit{ compatible } \sigma} \boldsymbol{P}[(\omega, \sigma)] \\
&= \frac{1}{Z^0_{G,p,q}} \sum_{\omega \textit{ compatible } \sigma} p^{o(\omega)}(1-p)^{c(\omega)} && \text{Eqn. 1} \\
&= \frac{1}{Z^0_{G,p,q}} \underbrace{(1-p)^{|E_\sigma|}}_{\text{deterministically closed}} \underbrace{\sum_{\omega' \in \{0,1\}^{E \setminus E_\sigma}} p^{o(\omega')}(1-p)^{c(\omega')}}_{\text{random}} \\
&= \frac{1}{Z^0_{G,p,q}} (1-p)^{|E_\sigma|} && \text{last sums to one} \\
&= \frac{1}{Z^0_{G,p,q}} e^{(-\frac{q}{q-1})|E_\sigma|},
\end{aligned}$$

where in the last passage we used the claimed relation between $(\beta, p)$. Now notice that the

Hamiltonian of compatible configurations takes form:

$$\begin{aligned}
H_G^{\mathrm{f}}[\sigma] &= -\sum_{xy\in E} \sigma_x \cdot \sigma_y && \text{Def. 1.15}\\
&= -\sum_{xy\in E_\sigma} \sigma_x \cdot \sigma_y + \sum_{xy\in E\setminus E_\sigma} \sigma_x \cdot \sigma_y \\
&= -\sum_{xy\in E_\sigma} -\frac{1}{q-1} - \sum_{xy\in E\setminus E_\sigma} 1 && \text{clever dot product}\\
&= \frac{1}{q-1}|E_\sigma| - |E \setminus E_\sigma| \\
&= \frac{q}{q-1}|E_\sigma| - |E|,
\end{aligned}$$

which eventually gives us:

$$\boldsymbol{P}[\sigma] = \underbrace{\frac{1}{Z^0_{G,p,q}} e^{-\beta|E|}}_{\perp\sigma} e^{-\beta H_G^{\mathrm{f}}[\sigma]} = \frac{1}{Z^{\mathrm{f}}_{G,\beta,q}} e^{-\beta H_G^{\mathrm{f}}[\sigma]},$$

where in the last passage we use the fact that a probability must sum to one.
A similar argument shows the second claim. Quickly notice that starting from a configuration $\sigma$ it holds that the joint is:

$$\boldsymbol{P}[(\sigma,\omega)] = \frac{e^{-\beta\cdot q|E_\sigma|} p^{o(\omega)}(1-p)^{|E|-o(\omega)-q|E_\sigma|}}{Z^{\mathrm{f}}_{G,\beta,q}} = \frac{p^{o(\omega)}(1-p)^{c(\omega)}}{Z^{\mathrm{f}}_{G,\beta,q}},$$

where we used the claimed relation between $(\beta, q)$ in the second equation. □

**Observation 2.8.** *For free, we also obtained the relation between partition functions:*

$$Z^0_{G,p,q} = e^{-\beta|E|} Z^{\mathrm{f}}_{G,\beta,q}.$$

For boundary conditions we instead have that the construction is the same but at the boundary we fix the spins to be $\sigma_{\mathcal{C}} = b$ in the first case and force the connection in the second case.
With similar arguments, we obtain the proposition in the wired case[4].

**Proposition 2.9** (Coupling for Monochromatic BC)**.** *Formulated as in [Duminil-Copin, 2017b].*

$$\begin{aligned}
&\texttt{percolation to spins} \quad \omega \sim \phi^1_{G,p,q} \implies \sigma \sim \mu^{\mathrm{b}}_{G,\beta,q}, \quad \beta := -\frac{q-1}{q}\ln(1-p),\\
&\texttt{spins to percolation} \quad \widetilde{\sigma} \sim \mu^{\mathrm{b}}_{G,\beta,q} \implies \widetilde{\omega} \sim \phi^1_{G,p,q}.
\end{aligned}$$

[4]aka monochromatic boundary conditions, where the only color is associated to the $b$ and is used in visualizations.

## 2.2 On the advantage of transferring results

The property just presented allows us to directly relate critical behaviors of the RCM and Potts models. Precisely, we have the following connection.

**Corollary 2.10** (Correlation is connection)**.** *Formulated as in [Duminil-Copin, 2017b]. Let $G \subset \mathbb{Z}^d$ finite, consider $q \in \mathbb{N}, q \geq 2, p \in [0,1], \beta > 0$. Assume $(p, \beta)$ are related by the formula of the propositions above. For any $x \in V$:*

$$\mu^{\mathrm{f}}_{G,\beta,q}[\sigma_x \cdot \sigma_y] = \phi^0_{G,p,q}[x \longleftrightarrow y],$$
$$\mu^{\mathrm{b}}_{G,\beta,q}[\sigma_x \cdot b] = \phi^1_{G,p,q}[x \longleftrightarrow \partial G].$$

*Proof.* Again, the proof is done for the free conditions only. Denote the coupling with $\boldsymbol{P}$ as above. Its expectation is $\boldsymbol{E}$. Then:

$$\begin{aligned}\mu^{\mathrm{f}}_{G,\beta,q}[\sigma_x \cdot \sigma_y] &= \boldsymbol{P}[\sigma_x \cdot \sigma_y, \omega \in \{0,1\}^E] \\ &= \boldsymbol{E}\left[(\sigma_x \cdot \sigma_y)(\mathbb{1}_{\{x \longleftrightarrow y\}} + \mathbb{1}_{\neg\{x \longleftrightarrow y\}})\right] \\ &= \boldsymbol{E}\left[(\sigma_x \cdot \sigma_y)\mathbb{1}_{\{x \longleftrightarrow y\}}\right] + \boldsymbol{E}\left[(\sigma_x \cdot \sigma_y)\mathbb{1}_{\neg\{x \longleftrightarrow y\}}\right],\end{aligned}$$

where by construction if $\{x \longleftrightarrow y\}$ there is perfect correlation and the two are equal, while if $\neg\{x \longleftrightarrow y\}$ there is total independence and correlation is null. Thus, the second term is null and expressing the dot product in the first we eventually get:

$$\mu^{\mathrm{f}}_{G,\beta,q}[\sigma_x \cdot \sigma_y] = \boldsymbol{E}\left[(\sigma_x \cdot \sigma_y)\mathbb{1}_{\{x \longleftrightarrow y\}}\right] = \boldsymbol{E}\left[\mathbb{1}_{\{x \longleftrightarrow y\}}\right] = \boldsymbol{P}[\{x \longleftrightarrow y\}] = \phi^0_{G,p,q}[x \longleftrightarrow y].$$

□

Apart from this, we can exploit the equivalence between correlation of spins and connection via edges to derive some very important results, which are in the end Duminil-Copin's contribution. It is worth remarking that while we have worked with finite graphs, the notions extend to infinite graphs and allow to couple criticalities of parameters.

# 3 One for All, All for One

Unus pro omnibus, omnes pro uno

(Unofficial) motto of Switzerland

Tous pour un, un pour tous

Alexandre Dumas, The Three Musketeers (1844)

Having recovered Potts models in the RCM via a coupling argument, a brief exposition of some consequences is presented.

Subject to additional lenghty and technical discussion this framework allows to recover many results. Below are the most direct ones. The theory in Duminil-Copin [2017b] until Section 3 is sufficient for them.
A famous and surprisingly articulate Theorem is that of the critical probability for percolation in the square lattice.

**Theorem 3.1** (Kesten's Theorem)**.** *First found in [Kesten, 1980].*
*For Bernoulli percolation on* $\mathbb{Z}^2$*, we have* $p_c = \frac{1}{2}$*, and* $\theta(p_c) = 0$*.*

In a wider sense, the contributions of Duminil-Copin and his group led to a quicker and new proof of the upper bound $p_c \geq \frac{1}{2}$, by means of a quantity:

$$\varphi_p(S) \coloneqq p \sum_{xy \in \Delta S} \mathbb{P}_p[0 \overset{S}{\longleftrightarrow} x],$$

which is in the Ising model view:

$$\varphi_\beta(S) \coloneqq \sum_{xy \in \Delta S} (1 - e^{-\beta J_{x,y}}) \mathbb{P}_\beta[0 \overset{S}{\longleftrightarrow} x] \quad p = e^{-\beta J_{x,y}},$$

allowing for a generalization of the sharpness result to the latter, via the coupling obtained with the RCM.
It turns out that an even more general Theorem can be obtained, which is redundant for showing the criticality.

**Theorem 3.2** (Supercritical and subcritical bounds)**.** *First appeareance in [Duminil-Copin et al., 2018], version of [Duminil-Copin, 2017b] Thm. 2.12.*
*The RCM on* $\mathbb{Z}^d$ *with* $q \geq 1$ *(not necessarily an integer) is such that:*

*(supercritical)* $\exists c > 0$ *s.t. for* $p > p_c$ *it holds* $\phi^1_{p,q}[0 \longleftrightarrow \infty] \geq c(p - p_c)$*, aka the mean field lower bound.*

*(subcritical) For* $p < p_c$ $\exists c_p > 0$ *s.t.* $\forall n \geq 1$ *it holds* $\phi^1_{\Lambda_n,p,q}[0 \longleftrightarrow \partial\Lambda_n] \leq e^{-c_p n}$*.*

*Remark* 3.3. The result is valid for any locally quasi-transitive graph $\mathbb{G}$.

Lastly, it is also possible to evaluate the critical probability of the RCM for any cluster weight on the square lattice, and not only for $q = 1$ (percolation) or $q = 2$ (Ising). This is possible by a self-duality argument similar to the simple case but extended along the way paved by the RCM.

**Theorem 3.4** (Critical parameter for square lattice of RCM)**.** *First appeareance in [Beffara and Duminil-Copin, 2013], version of [Duminil-Copin, 2017b] Thm. 2.15.*
*Consider the RCM on* $\mathbb{Z}^2$ *with* $q \geq 1$*. Then:*

$$p_c = \frac{\sqrt{q}}{1 + \sqrt{q}},$$

*and for* $p < p_c$ *we have that* $\exists c_p > 0$ *such that* $\phi^1_{p,q}[0 \longleftrightarrow \partial\Lambda_n] \leq e^{-c_p n}$ *for any* $n \geq 0$*.*

Eventually a side argument returns also the critical temperature of the RCM.

**Corollary 3.5** (Critical Temperature square lattice RCM)**.** *First appeareance in [Beffara and Duminil-Copin, 2013], version of [Duminil-Copin, 2017b] Cor. 2.16.*
*We have:*

$$\beta_c(q) \coloneqq \frac{q-1}{q} \log[1+\sqrt{q}].$$

# References


Vincent Beffara and Hugo Duminil-Copin. The self-dual point of the two-dimensional random-cluster model is critical for $q\geq 1$, November 2013.

Hugo Duminil-Copin. *Phase Transition in Random-cluster and O(n)-models*. PhD thesis, 2011.

Hugo Duminil-Copin. Introduction to Bernoulli percolation. 2017a.

Hugo Duminil-Copin. Lectures on the Ising and Potts models on the hypercubic lattice, July 2017b.

Hugo Duminil-Copin. Sixty years of percolation, December 2017c.

Hugo Duminil-Copin and Vincent Tassion. A New Proof of the Sharpness of the Phase Transition for Bernoulli Percolation and the Ising Model. *Communications in Mathematical Physics*, 343(2):725–745, April 2016. ISSN 1432-0916. doi: 10.1007/s00220-015-2480-z.

Hugo Duminil-Copin and Vincent Tassion. A new proof of the sharpness of the phase transition for Bernoulli percolation on Zd. 2017.

Hugo Duminil-Copin, Aran Raoufi, and Vincent Tassion. Sharp phase transition for the random-cluster and Potts models via decision trees, December 2018.

Geoffrey Grimmett. *Percolation*, volume 321 of *Grundlehren Der Mathematischen Wissenschaften*. Springer Berlin Heidelberg, Berlin, Heidelberg, 1999. ISBN 978-3-642-08442-3 978-3-662-03981-6. doi: 10.1007/978-3-662-03981-6.

Harry Kesten. The critical probability of bond percolation on the square lattice equals 1/2. *Communications in Mathematical Physics*, 74(1):41–59, February 1980. ISSN 1432-0916. doi: 10.1007/BF01197577.

Jhih-Huang Li. *Universality of the Random-Cluster Model and Applications to Quantum Systems*. PhD thesis, 2017.