A view on Percolation and Spin Systems

Accompanying Images

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Lecture Contents

1. Percolation vs Spin Systems
2. Random Cluster Model
3. One for All, All for One

Quote Slide

A mathematician is a device for turning coffee into theorems.

Alfréd Rényi, often ascribed to Paul Erdős

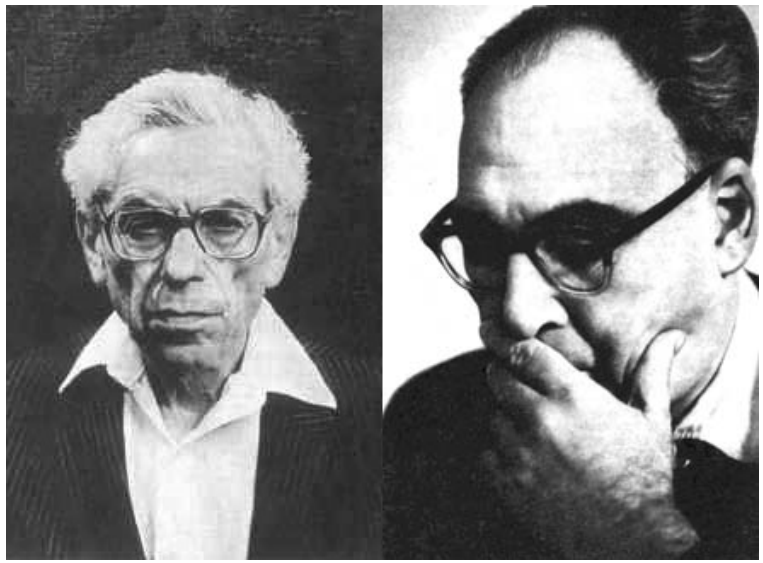

Figure: Paul Erdős and Alfréd Rényi

Preamble

Sources Partial extraction of the framework of H. Duminil-Copin, 2022 Fields Medalist, plus side material. More or less in order of technicality [Dum17a; Dum17b; Dum17c; DT16; Gri99; Li17; Dum11; DT17]. The first reading [Dum17a] is exhaustive for a complete reference structure.

Aims Draw connections between a probabilistic-graphical and a Statistical Mechanics model.
Structure given the strong mathematical structure of the results, only intuitions and some proofs. Details are in the references!

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A motivating Example

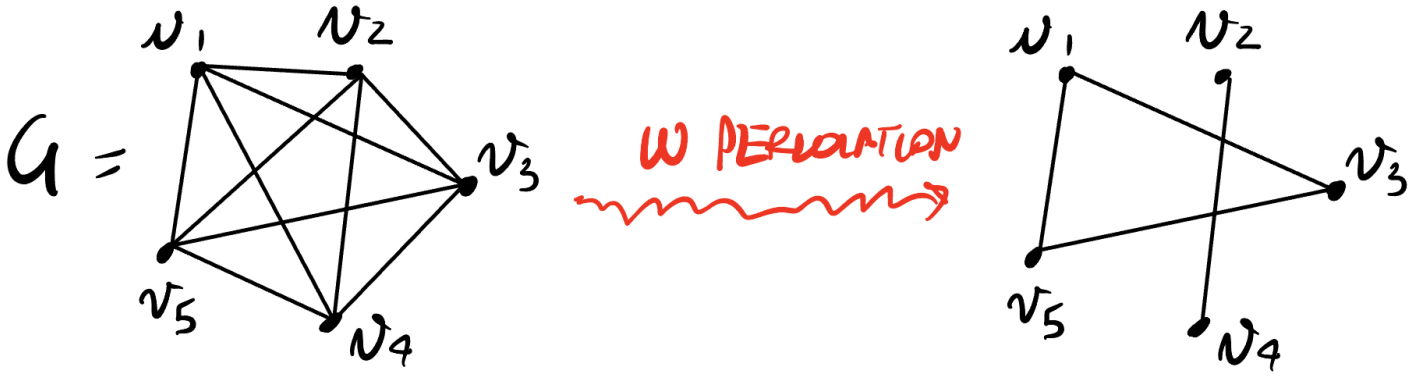

Figure: Anti-percolation

Percolation Simulation

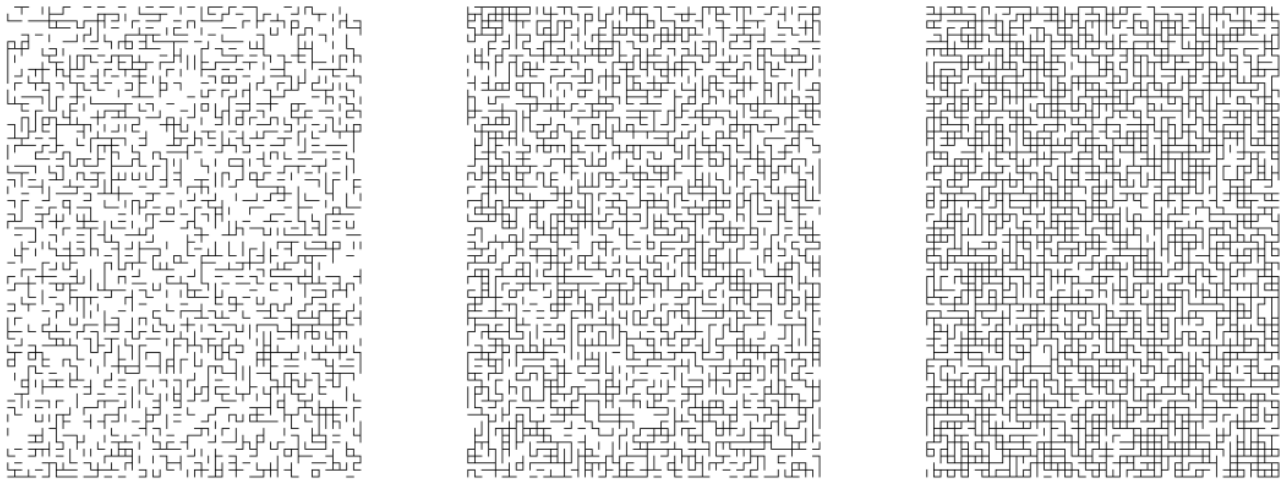

Figure: Percolation simulation at $p < \frac{1}{2}, p = \frac{1}{2}, p > \frac{1}{2}$ in the square lattice. Source [Li17]

Duality of graphs

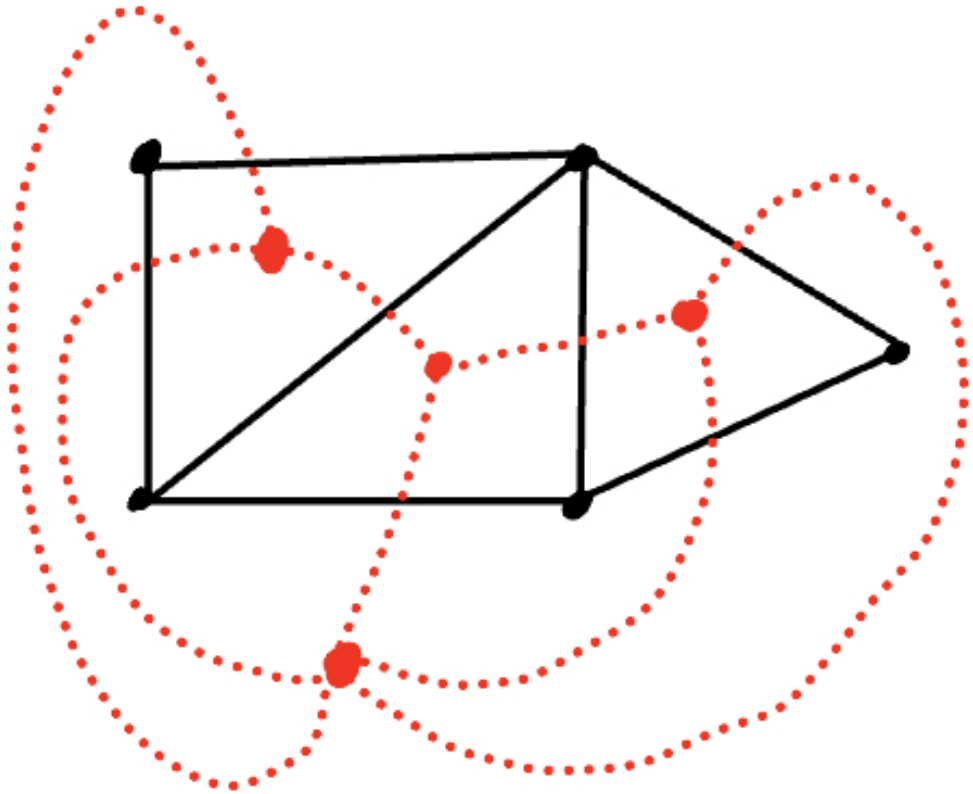

Figure: A dual graph

Duality of Lattices

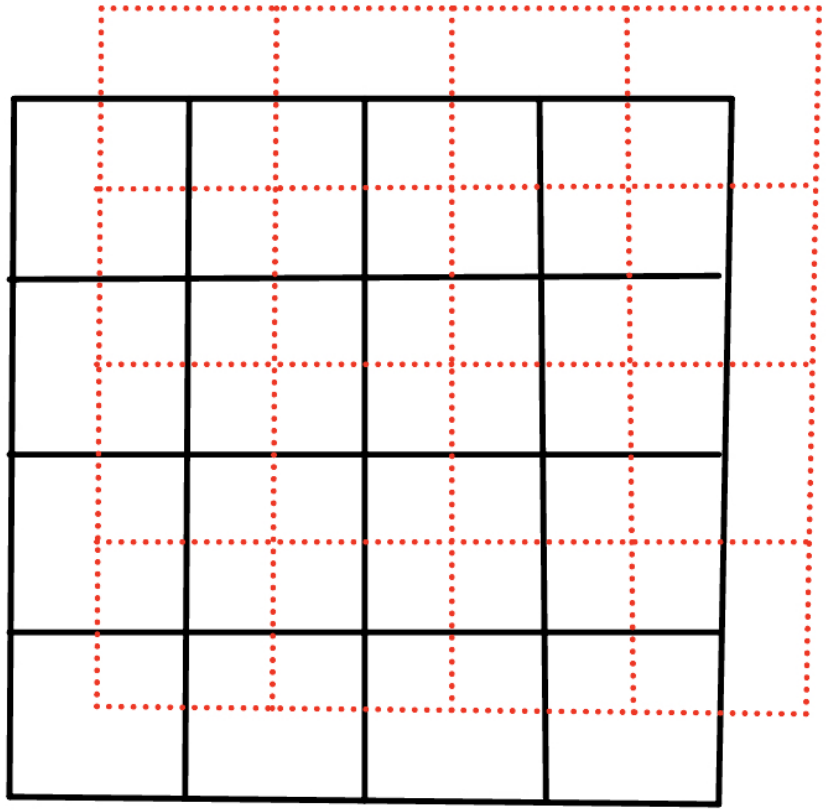

Figure: Dual subgrid of $\mathbb{Z}^2$

Duality of percolation

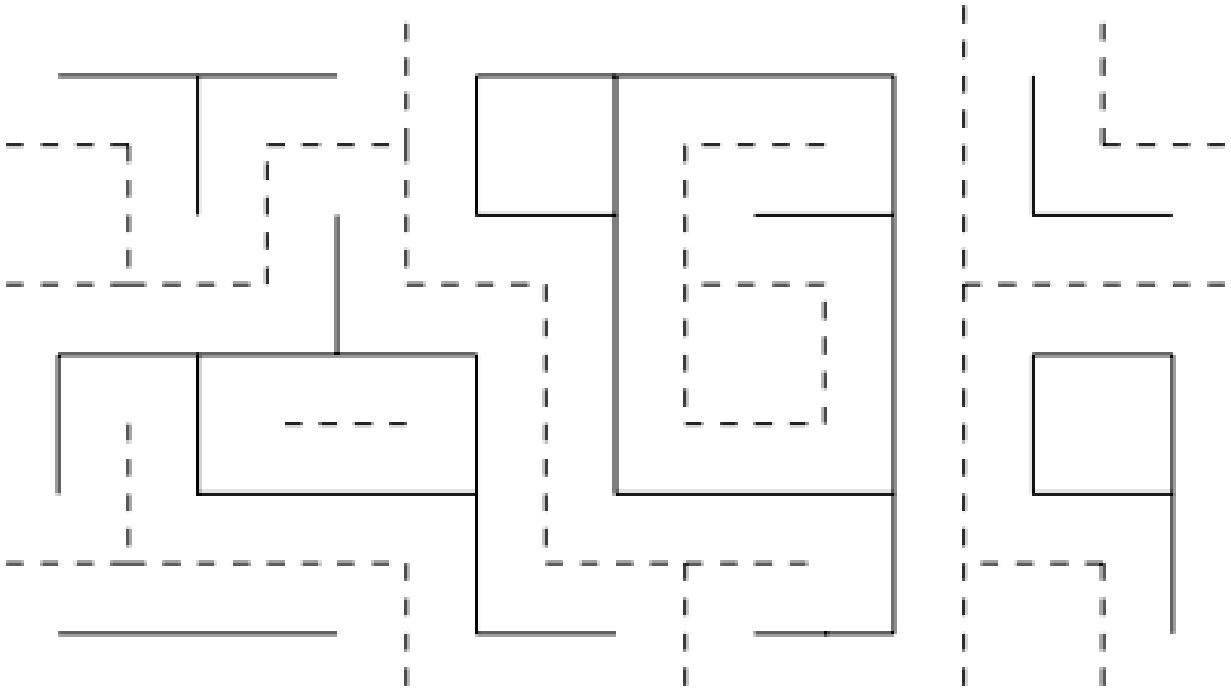

Figure: Primal and dual percolation. Source [Dum17b]

Potts coloring

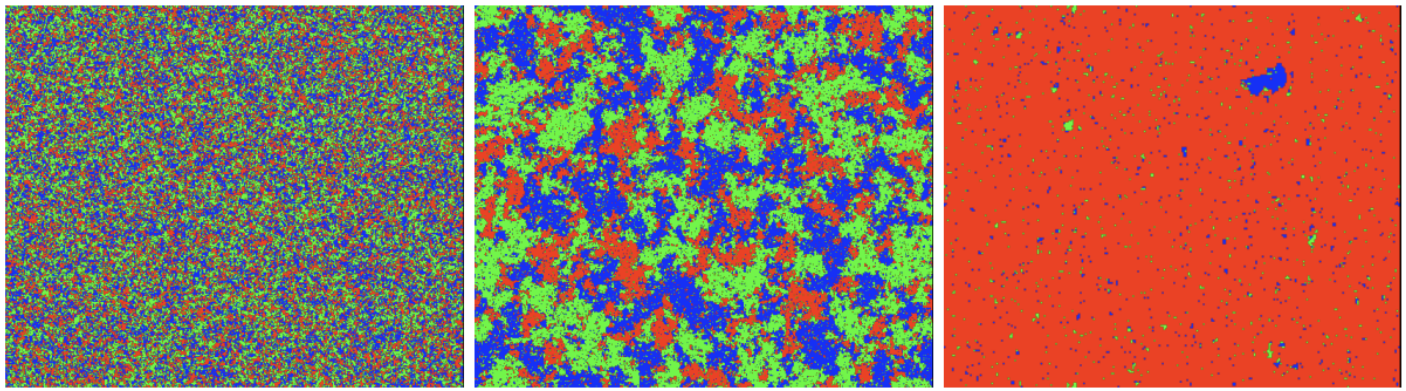

Figure: Simulations for $q = 3, d = 2$ at subcritical, critical, supercritical temperature. Source [Dum17c]

Critical Simulation

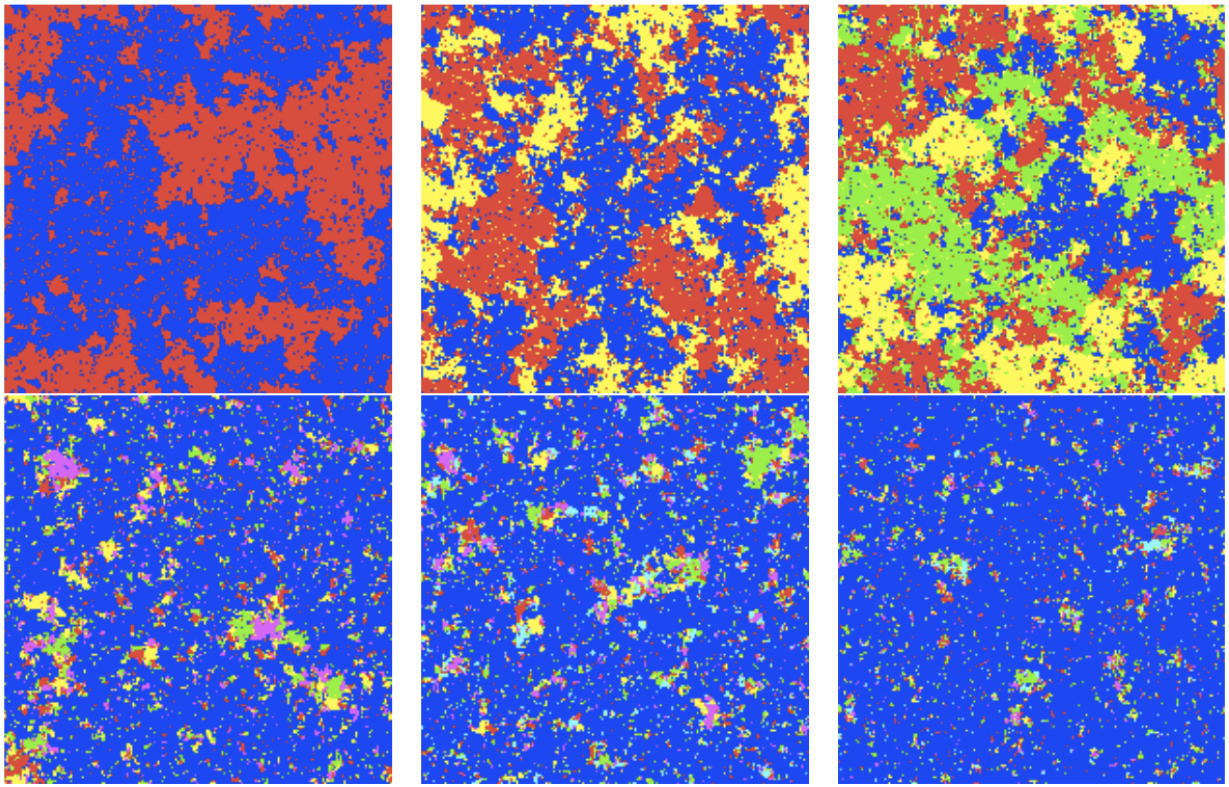

Figure: Simulations for $q \in \{2, 3, 4, 5, 6, 9\}$, $d = 2$ at criticality. Source [Dum17c]

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Coupling

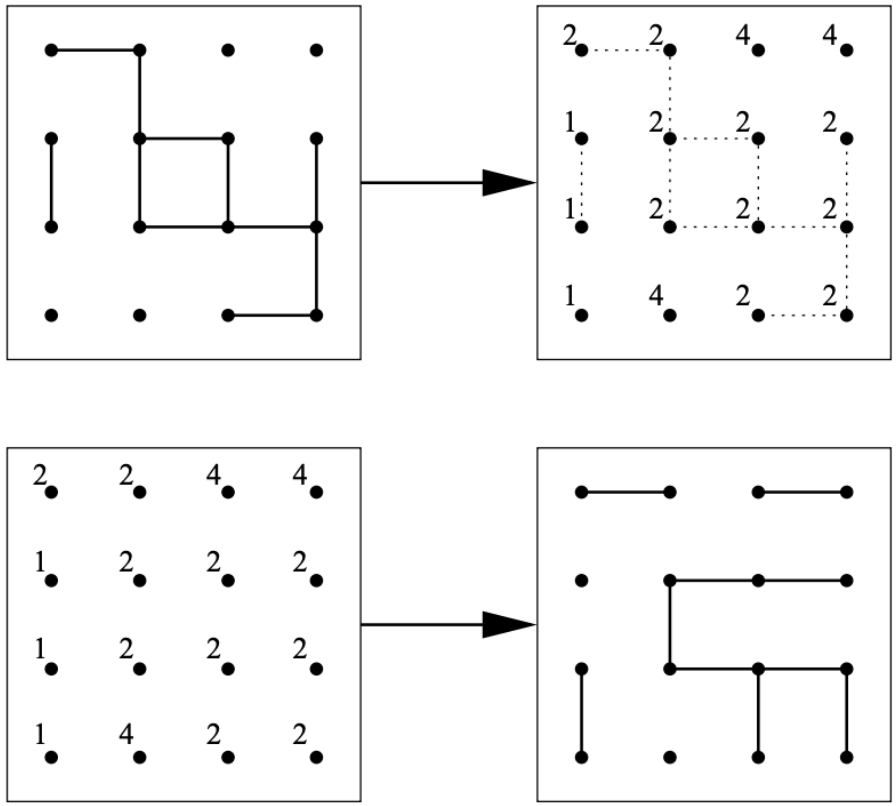

Figure: The methods for $q = 4$. Source [Gri99]

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Quote Slide

Unus pro omnibus, omnes pro uno – Tous pour un, un pour tous

(Unofficial) motto of Switzerland – Alexandre Dumas, The Three Musketeers (1844)

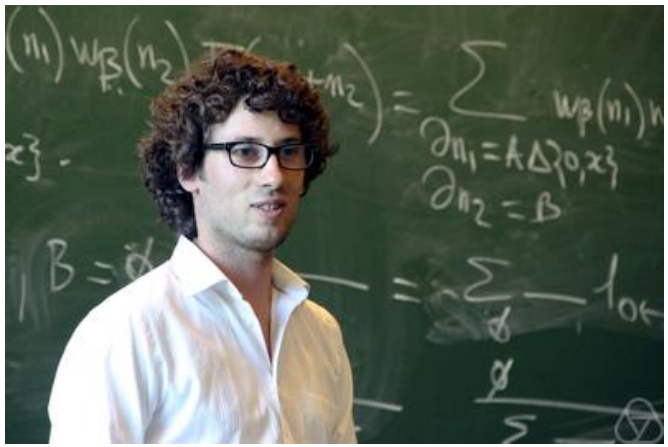

Figure: Hugo Duminil-Copin

Concluding

Any question/discussion, let me know!

Thank you!

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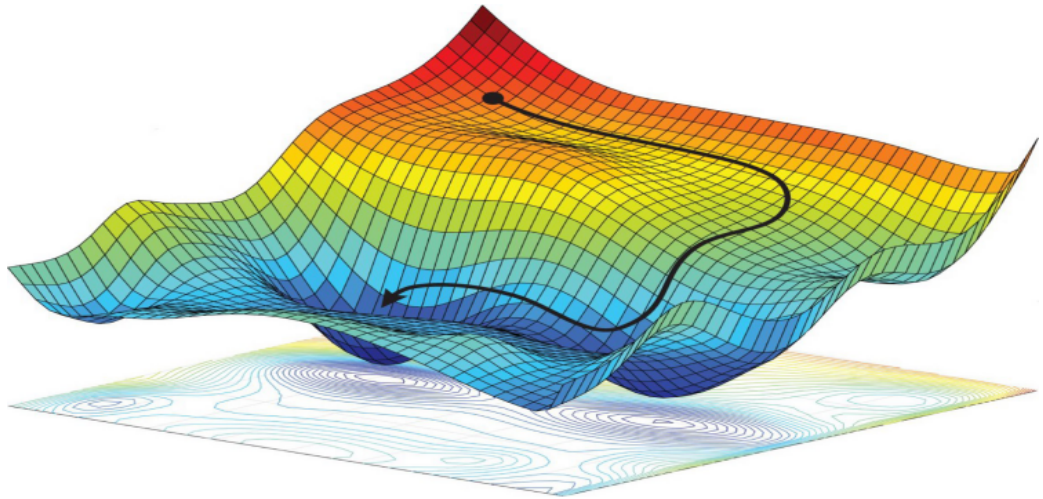

References I

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[Dum17b] Hugo Duminil-Copin. "Introduction to Bernoulli Percolation". 2017.

[Dum17c] Hugo Duminil-Copin. *Lectures on the Ising and Potts Models on the Hypercubic Lattice*. July 2017. arXiv: arXiv:1707.00520. (Visited on 03/17/2023).

[DT16] Hugo Duminil-Copin and Vincent Tassion. "A New Proof of the Sharpness of the Phase Transition for Bernoulli Percolation and the Ising Model". In: *Communications in Mathematical Physics* 343.2 (Apr. 2016), pp. 725–745. ISSN: 1432-0916. DOI: 10.1007/s00220-015-2480-z. (Visited on 03/17/2023).

References II

[Gri99] Geoffrey Grimmett. *Percolation*. Ed. by S. S. Chern et al. Vol. 321. Grundlehren Der Mathematischen Wissenschaften. Berlin, Heidelberg: Springer Berlin Heidelberg, 1999. ISBN: 978-3-642-08442-3 978-3-662-03981-6. DOI: 10.1007/978-3-662-03981-6. (Visited on 03/20/2023).

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[Dum11] Hugo Duminil-Copin. "Phase Transition in Random-cluster and O(n)-models". PhD thesis. 2011.

[DT17] Hugo Duminil-Copin and Vincent Tassion. "A New Proof of the Sharpness of the Phase Transition for Bernoulli Percolation on Zd". In: (2017).